separation point with rotation rate is predicted. This result is in agreement with the limited experimental data available[7,8] and a theoretical analysis.[14]

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Orthogonality of Generally Normalized Eigenvectors and Eigenrows

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FREE vibration analysis of linear dynamic systems in the presence of viscous damping leads to the generalized eigenvalue problem

$$Q(\lambda)x \equiv [\lambda^2 A + \lambda B + C]x = 0 \qquad (1)$$

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In this equation the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ represent the inertia, damping and stiffness properties, respectively, of the system. They are of order $n \times n$ where n is the number of degrees of freedom considered in the analysis.

Equation (1) has, in general, $2n$ eigenvalues λ_r associated with $2n$ eigenvectors $\boldsymbol{x}_r$ and $2n$ eigenrows $\boldsymbol{y}_s^T$.[1] They satisfy the equations

$$[\lambda_r^2 \boldsymbol{A} + \lambda_r \boldsymbol{B} + \boldsymbol{C}]\boldsymbol{x}_r = 0, \quad (r = 1,2,\ldots,2n) \qquad (2)$$

and

$$\boldsymbol{y}_s^T [\lambda_s^2 \boldsymbol{A} + \lambda_s \boldsymbol{B} + \boldsymbol{C}] = 0, \quad (s = 1,2,\ldots,2n) \qquad (3)$$

If the eigenvalues are all distinct, each eigenvector and eigenrow is uniquely defined to the extent of an arbitrary multiplier which is chosen to satisfy a convenient normalization criterion.

The eigenvectors and eigenrows are also orthogonal in the sense that, for $r \neq s$,

$$\boldsymbol{y}_s^T [(\lambda_s + \lambda_r)\boldsymbol{A} + \boldsymbol{B}]\boldsymbol{x}_r = 0, \ (r,s = 1,2,\ldots,2n) \qquad (4)$$

For $r = s$, however, the left-hand side of Eq. (4) does not vanish, and it can be made equal to any desired value by adjusting the arbitrary multipliers of the normalized $\boldsymbol{x}_r$ and $\boldsymbol{y}_r^T$. When normalization is done according to the criterion

$$\boldsymbol{y}_r^T [2\lambda_r \boldsymbol{A} + \boldsymbol{B}]\boldsymbol{x}_r = 1, \ (r = 1,2,\ldots,2n) \qquad (5)$$

Eqs. (4) and (5) can be combined in the single matrix equation

$$\Lambda YAX + YAX\Lambda + YBX = I \qquad (6)$$

where

$$X = \begin{bmatrix} x_1 & x_2 & \ldots & x_{2n} \end{bmatrix} \qquad (7a)$$

$$Y = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_{2n}^T \end{bmatrix} \qquad (7b)$$

$$\Lambda = \text{diag}\ \{\lambda_1, \lambda_2, \ldots, \lambda_{2n}\} \qquad (7c)$$

and

I is the unit matrix of order $2n$ (7d)

Under this particular normalization, it has been found[2] that each row of the eigenvectors modal matrix X is orthogonal to each column of the eigenrows modal matrix Y, that is to say

$$XY = 0 \qquad (8)$$

Equation (8) is an interesting result which has found some useful applications in the theory of forced vibration of non-conservative systems.[3] Its validity relies, of course, on the adoption of Eq. (5) as a criterion for normalization. But in some cases, Eq. (5) is not very convenient, and other criteria are preferred. The question then arises as to whether an orthogonality relationship can still be found to replace Eq. (8) in such a case. This question is answered here by deriving the general orthogonality relationship corresponding to any normalization criterion, including that of Eq. (5).

The only influence of the choice of normalization criterion is to change the right-hand side of Eq. (5) from unity to any other constant m_r. Equations (5) and (6) then take, respectively, the forms

$$y_r^T[2\lambda_r A+B]x_r=m_r, \quad (r=1,2,\ldots,2n) \tag{9}$$

and

$$\Lambda YAX+YAX\Lambda+YBX=M \tag{10}$$

in which

$$M=\text{diag}\{m_1,m_2,\ldots,m_{2n}\} \tag{11}$$

It will now be shown here that the matrices X and Y which satisfy Eq. (10) are related in such a way that

$$XM^{-1}Y=0 \tag{12}$$

i.e., each row of the eigenvectors modal matrix X and column of the eigenrows modal matrix Y are orthogonal relative to the inverse of the normalization matrix M.[3]

To prove Eq. (12), we start by considering Eq. (1). For distinct eigenvalues λ_r $(r=1,2,\ldots,2n)$ the matrix

$$Q(\lambda_r)=[\lambda_r^2A+\lambda_rB+C] \tag{13}$$

is simply degenerate (of rank $n-1$) and hence its adjoint has degeneracy $n-1$ (or unit rank). Consequently, (Ref. 4, p. 61), there must exist two nonzero vectors u_r and v_r such that

$$\text{adj}Q(\lambda_r)=u_rv_r^T \tag{14}$$

It therefore follows that

$$Q(\lambda_r)\text{adj}Q(\lambda_r)=Q(\lambda_r)u_rv_r^T \tag{15a}$$

and

$$\text{adj}Q(\lambda_r)Q(\lambda_r)=u_rv_r^TQ(\lambda_r) \tag{15b}$$

But each matrix commutes with its adjoint so that[4]

$$Q(\lambda)\text{adj}Q(\lambda)=\text{adj}Q(\lambda)Q(\lambda)=\det Q(\lambda)I \tag{16}$$

which, at $\lambda=\lambda_r$, reduces to

$$Q(\lambda_r)\text{adj}Q(\lambda_r)=\text{adj}Q(\lambda_r)\ Q(\lambda r)=\det Q(\lambda_r)I=0 \tag{17}$$

Since v_r^T and u_r are nonzero, Eqs. (15) are reduced by Eqs. (17) to

$$Q(\lambda_r)u_r=0 \tag{18a}$$

and

$$v_r^TQ(\lambda_r)=0 \tag{18b}$$

Equations (18) show that u_r and v_r^T are eigenvectors and eigenrows, respectively, for the matrix $Q(\lambda_r)$. Hence they are related to x_r and y_r^T through the equations

$$x_r=\alpha_ru_r \tag{19a}$$

and

$$y_r^T=\beta_rv_r^T, \quad (r=1,2,\ldots,2n) \tag{19b}$$

where α_r and β_r are scalar multipliers.

The vectors u_r and v_r are also related to the first derivative of the determinant of $Q(\lambda)$ with respect to λ at $\lambda=\lambda_r$. This is denoted by $\delta'(\lambda_r)$, and it is the sum of the determinants obtained by replacing each row, or column, of $\delta(\lambda)$ in turn by its derivative with respect to λ and then putting $\lambda=\lambda_r$ (see Ref. 4, p. 16). If e_r denotes a null vector (of order n) which has its rth element only replaced by 1; it can be written down as

$$\delta'(\lambda_r)=\left|\frac{d}{d\lambda}\det Q(\lambda)\right|_{\lambda=\lambda_r}=\sum_{r=1}^{n}[e_r^TQ'(\lambda_r)][\text{adj}Q(\lambda_r)e_r]$$

$$=\text{trace}Q'(\lambda_r)uv_r^T \ =v_r^TQ'(\lambda_r)u_r\,(r=1,2,\ldots,2n) \tag{20}$$

On substituting Eqs. (19) and the derivative of Eq. (13) into Eqs. (20) the result is

$$\delta'(\lambda_r)\alpha_r\beta_r=y_r^T[2\lambda_rA+B]x_r,\,(r=1,2,\ldots,2n) \tag{21}$$

Equations (9) and (21) show that the multipliers α_r and β_r must satisfy the relationships

$$m_r=\alpha_r\beta_r\delta'(\lambda_r),\,(r=1,2,\ldots,2n) \tag{22}$$

and it is to be noticed here that $\delta'(\lambda_r)$ cannot be zero in the case of distinct eigenvalues.

When use is made of Eqs. (19), (22), and (14), it is found is

$$XM^{-1}Y=\sum_{r=1}^{2n}\frac{1}{m_r}x_ry_r^T=\sum_{r=1}^{2n}\frac{1}{\delta'(\lambda_r)}\text{adj}\,Q(\lambda_r)$$

The ijth element of $XM^{-1}Y(i,j=1,2,\ldots,n)$ is therefore

$$(XM^{-1}Y)_{ij}=\sum_{r=1}^{2n}\frac{P_{ij}(\lambda_r)}{\delta'(\lambda_r)} \tag{23}$$

where $P_{ij}(\lambda_r)$ is the ijth element of adj $Q(\lambda_r)$. To evaluate the right-hand side of Eq. (23) we notice that the ijth element of adj $Q(\lambda)$, i.e. $P_{ij}(\lambda)$, is a polynomial of degree $(2n-2)$, at most, in λ, while det $Q(\lambda)$, i.e. $\delta(\lambda)$, is a polynomial of degree $2n$ in λ. Their quotient is therefore expressible by partial fraction theorem in the form

$$\frac{P_{ij}(\lambda)}{\delta(\lambda)}=\sum_{r=1}^{2n}\frac{K_r}{(\lambda-\lambda_r)} \tag{24}$$

where

$$K_r=\lim_{\lambda\to\lambda_r}\frac{(\lambda-\lambda_r)P_{ij}(\lambda)}{\delta(\lambda)}$$

$$=\lim_{\lambda\to\lambda_r}\frac{P_{ij}(\lambda)}{\delta'(\lambda)}$$

$$=\frac{P_{ij}(\lambda_r)}{\delta'(\lambda_r)} \tag{25}$$

Eqs. (24) and (25) show that

$$\frac{P_{ij}(\lambda)}{\delta(\lambda)}=\sum_{r=1}^{2n}\frac{1}{(\lambda-\lambda_r)}\frac{P_{ij}(\lambda_r)}{\delta'(\lambda_r)} \tag{26}$$

On multiplying Eq. (26) by λ and taking the limits on both sides as $\lambda\to\infty$ it reduces to

$$\lim_{\lambda\to\infty}\frac{\lambda P_{ij}(\lambda)}{\delta(\lambda)}=\sum_{r=1}^{2n}\frac{P_{ij}(\lambda_r)}{\delta'(\lambda_r)} \tag{27}$$

When Eq. (23) is now substituted into Eq. (27) the result is

$$(XM^{-1}Y)_{ij} = \lim_{\lambda \to \infty} \frac{\lambda P_{ij}(\lambda)}{\delta(\lambda)} \tag{28}$$

The right-hand side of this last equation vanishes since $P_{ij}(\lambda)$ is of order $2n-2$, and $\delta(\lambda)$ is of order $2n$ in λ. Hence the ijth element of $XM^{-1}Y$ vanishes (regardless of the values of i and j). Consequently the product $XM^{-1}Y$ is a null matrix, and Eq. (12) is proved.

It is to be noticed that the proof just given does not depend on any restrictions imposed on the matrices A, B, and C. The only requirement is that the system must have its $2n$ eigenvalues all distinct. Equation (12) is therefore applicable to nonconservative as well as conservative systems which meet this requirement. A numerical example is presented to illustrate these ideas.

Consider the free vibration of a two-degree-of-freedom nonconservative system whose equation is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -4 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 2 & -6 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{29}$$

Here the stiffness matrix is not symmetric, and the damping matrix is neither symmetric nor positve definite. On substituting a solution of the form $xe^{\lambda t}$, Eq. (29) yields the eigenvalue problem described by Eq. (1) where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -4 & 1 \\ 3 & 3 \end{bmatrix}, C = \begin{bmatrix} 2 & -6 \\ -1 & 6 \end{bmatrix} \tag{30}$$

The eigenvalues of the system are the roots of the characteristic equation

$$\det Q(\lambda) \equiv \begin{vmatrix} \lambda^2 - 4\lambda + 2 & \lambda - 6 \\ 3\lambda - 1 & \lambda^2 + 3\lambda + 6 \end{vmatrix} = 0 \tag{31}$$

and on expanding the determinant, they are found to be

$$\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = 3 \tag{32}$$

Corresponding to these eigenvalues, Eq. (31) shows that

$$\mathrm{adj}Q(\lambda_1) = \begin{bmatrix} 4 & 8 \\ 7 & 14 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix} [1 \quad 2] \tag{33a}$$

$$\mathrm{adj}Q(\lambda_2) = \begin{bmatrix} 4 & 7 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [4 \quad 7] \tag{33b}$$

$$\mathrm{adj}Q(\lambda_3) = \begin{bmatrix} 10 & 5 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} [2 \quad 1] \tag{33c}$$

$$\mathrm{adj}Q(\lambda_4) = \begin{bmatrix} 24 & 3 \\ -8 & -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} [8 \quad 1] \tag{33d}$$

The eigenvectors and eigenrows can now be taken as any multiples of the columns and rows, respectively, of the right-hand side of Eqs. (33). If they are normalized in such a way that the first element of each is always unity, we have

$$x_1 = \begin{bmatrix} 1 \\ 7/4 \end{bmatrix}; x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; x_3 = \begin{bmatrix} 1 \\ -1/5 \end{bmatrix}; x_4 = \begin{bmatrix} 1 \\ -1/3 \end{bmatrix} \tag{34a}$$

and

$$y_1^T = [1 \quad 2] \tag{34b}$$

$$y_2^T = [1 \quad 7/4] \tag{34c}$$

$$y_3^T = [1 \quad 1/2] \tag{34d}$$

$$y_4^T = [1 \quad 1/8] \tag{34e}$$

The normalization matrix in such a case is

$$\begin{aligned} M &= \mathrm{diag}\ \{ y_i^T (2\lambda_i A + B) x_i \}\ (i = 1,2,3,4) \\ &= \mathrm{diag}\ \{ -15/4, 2, -6/5, 5/3 \} \end{aligned} \tag{35}$$

Equations (34) and (35) readily show that

$$XM^{-1}Y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 7/4 & 1 & -1/5 & -1/3 \end{bmatrix} \begin{bmatrix} -4/15 & & & \\ & 1/2 & & \\ & & -5/6 & \\ & & & 3/5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 7/4 \\ 1 & 1/2 \\ 1 & 1/8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{36}$$

Equation (36) verifies the result predicted by Eq. (12).

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Integral Equation Methods in Duct Acoustics for Nonuniform Ducts with Variable Impedance

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Introduction

TO handle nonuniform geometries and/or nonuniform wall linings one can go immediately to a numerical solution, as in the finite-difference method,[1-3] or one can

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